Unstable Periodic Orbits as a Unifying Principle in the Presentation of Dynamical Systems in the Undergraduate Physics Curriculum

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Periodic orbits are familiar...

- Gravitational two-body problem (Newton, 1687)
- Three-body problem (Euler, Lagrange, Jacobi)
- Gave rise to our appreciation of chaotic orbits (Poincaré)
...but still a source of new results!

- New orbits found numerically (Moore, 1994)
- New orbits existence proven (Chenciner, Montgomery, 2001)
Lessons learned

- Some history of science
- Students may be invited to consider the difference between configuration space ($\mathbb{R}^3$), and phase space ($\mathbb{R}^6$ for two-body problem and $\mathbb{R}^9$ for three-body problem).
- Reduction of dimensionality of phase space when constants of the motion are known (e.g., COM and relative coordinates)
- Students see that this is still a vibrant and active area of research.
General Dynamical Systems

- Contrast dynamical systems in physics and mathematics
- State space $M$
- Evolution rule $\Phi_t : M \rightarrow M$
- Time $t$
  - Continuous time - differential equation - flow
  - Discrete time - difference equation - map
- Physics students will have seen Newton’s equations
- Prior course in differential equations
- Stretch for them to see relationship to simple maps
- Too simple to be useful?
Example: The Bernoulli map

- Discrete time dynamical system on the state space $[0, 1]$
- Dynamical equation: $x_{n+1} = 2x_n \mod 1$
- Example periodic orbit: $\frac{1}{3} \leftrightarrow \frac{2}{3}$
- Unstable: $0.33 \to 0.66 \to 0.32 \to 0.64 \to \cdots$ (Yorke)
- Map shifts base-two decimal point to the right
- Any rational number is a periodic point
- Rational numbers countable and dense in $[0, 1]$
- UPOs can be ordered by period
- Irrational numbers give chaotic trajectories
### Example: Bernoulli map UPOs

| Period $T_p$ | Lyndon word $L_p$ | Initial condition $x_0$ | Observables 
$\sum_{j=0}^{T_p-1} x_j$ | $\sum_{j=0}^{T_p-1} x_j^2$ |
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Lessons learned

- Very simple dynamical systems can exhibit both periodicity and chaos
- State space is generally replete with UPOs
State space is again $[0, 1]$

One-parameter family of maps:

$$x_{n+1} = f_\lambda(x_n) := 4\lambda x_n (1 - x_n).$$

Pitchfork bifurcation, stable and unstable periodic orbits

Easy programming exercise for students
Finding period-two orbits:

- Solve: \( x = f_\lambda(f_\lambda(x)) \)
- Minimize: \( F(x, y) = [x - f_\lambda(y)]^2 + [y - f_\lambda(x)]^2 \)

Finding period-three orbits, etc:

- Solve: \( x = f_\lambda(f_\lambda(f_\lambda(x))) \)
- Minimize: 
  \[
  F(x, y, z) = [x - f_\lambda(y)]^2 + [y - f_\lambda(z)]^2 + [z - f_\lambda(x)]^2
  \]
Logistic map with $\lambda > 1$

Interval leaves state space in one iteration

Preimages of interval leave in two iterations, etc.
Lessons Learned

- The notions of transient behavior and attracting set
- Period doubling route to chaos
- Pitchfork bifurcation
- UPOs discovered by either root finding or minimization
- Cantor-set nature of state space when $\lambda > 1$
Lorenz attractor

- Continuous time dynamical system on the state space $\mathbb{R}^3$
- Dynamical equations:

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + Rx - y \\
\dot{z} &= xy - bz
\end{align*}
\]

- Attracting set has periodic orbits
Unstable Periodic Orbits of Lorenz Attractor

- Correspond to any binary sequence (e.g., 11110)
- Dense in the attractor
- System is *hyperbolic*

If you know all UPOs with period $< T$, you can make statistical predictions of any observable (DZF formalism).
- UPOs and their properties can be tabulated, stored, and made available in a curated database.
These can be tabulated...


- Tabulated up to $\sim 20$ symbols
Lessons Learned

- These observations work for continuous-time dynamical systems
- The same labeling of orbits used in the Bernoulli map works for the Lorenz attractor
- Symbolic dynamics
Laminar and periodic flow

**Stable periodic orbits**

- Laminar (stationary) flow is a fixed point in function space
- von Kármán vortex street is a closed periodic orbit in function space
Turbulent flow in two dimensions


- Periodic flow in square domain
- Periodic force
  \[ \mathbf{F} = A \sin(2\pi mx) \sin(2\pi ny) \mathbf{e}_x + A \cos(2\pi mx) \cos(2\pi ny) \mathbf{e}_y \]
- Flow closely follows \( \mathbf{F} \) for low Re
- Turbulent for high Re
Lessons Learned

- All this can be made to work for dynamical systems on infinite-dimensional state-spaces
Unstable Periodic Orbits (UPOs)

- Attracting sets in a wide variety of dynamical systems are replete with periodic orbits
- If the dynamics are hyperbolic, the UPOs are unstable
- The UPOs are dense in the attracting set
- The UPOs are countable and have measure zero in the attracting set
- In spite of zero measure, UPOs are exceedingly important, as averages over the natural measure can be derived from them
- “The skeleton of chaos” (Cvitanovic)
Attracting sets and turbulent averages

- The driven Navier-Stokes equations in the turbulent regime describe nonlinear dynamics in an infinite-dimensional (function) space.
- These dynamics possess an attracting set.
- The attracting set is finite-dimensional, and its dimension grows as a power law in Reynolds number (Constantin, Foias, Manley, Temam, 1985).
- Long-time averages over this attracting set impart a “natural measure” to it.
- The problem of turbulence is that of extracting averages of observables over this natural measure.
Computing UPOs I: Shooting Method

- Begin on surface of codimension one in function space
- Evolve NS equations until return to that surface
- Use Newton-Raphson to close the gap
- Serial in time
- Constructed to obey equations of motion, but not periodic
Computing UPOs II: Relaxation Method

- Begin with periodic orbit that is smooth
- Relax to solution of NS equations using variational principle

\[ \Delta([f], T) = \frac{1}{2} \sum_{t=0}^{T-1} \sum_r \sum_j \left| f_j(r, t + 1) - f_j(r, t) - \frac{1}{\tau} \left[ f_{eq}^j(r, t) - f_j(r, t) \right] \right|^2 \]

- Constructed to be periodic, but not obey equations of motion
- Conjugate-gradient algorithm
- Higher-order differencing needed
- Local spline fitting to orbit needed
- *Enormous* amounts of memory are needed
Work in Progress I

- General HPC tools
- Clustering alg.
  - HMM
  - Buhl–Kennel
- Initial guess for UPOs
- Time sequence
- Plot of $\Delta(t,T)$
- Conjugate gradient
- Symbolic dynamics
- Kennel–Mees
- UPOs
- Improved accuracy
- Monodromy matrix
- Statistical error $\sim N^{-1/2}$
- Orr–Sommerfeld equation
- Pseudospectra
- Dimension of attractor
- Turbulent averages
- Improved accuracy
- $\zeta$ function
- Hydrodynamics
- Computation
- Current and future work
Work in Progress II

- LUPO: Laboratory for unstable periodic orbits
- Shell models of turbulence (Tang, Boghosian)
- 2D Navier-Stokes (Lätt, Smith, Boghosian)
- 3D Navier-Stokes (Faizendeiro, Coveney, Boghosian)
Conclusions

- UPOs are a unifying concept in dynamical systems
- Connects mathematical and physical way of understanding dynamical systems
- Improved understanding of fluid UPOs may lead to new statistical descriptions of turbulence
- HPC is just at the point where this can be done for infinite-dimensional systems
- LUPO will give students a way to experiment with such systems
- Creation of a UPO database will help spread and share information about UPOs