Unstable Periodic Orbits as a Unifying Principle in the Presentation of Dynamical Systems in the Undergraduate Physics Curriculum

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   - General HPC tools
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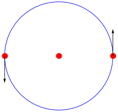
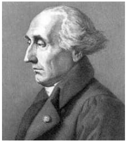
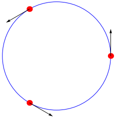
   - Conclusions

Periodic orbits are familiar...

- Gravitational two-body problem (Newton, 1687)
- Three-body problem (Euler, Lagrange, Jacobi)

- Gave rise to our appreciation of chaotic orbits (Poincaré)

...but still a source of new results!

- New orbits found numerically (Moore, 1994)
- New orbits existence proven (Chenciner, Montgomery, 2001)

Lessons learned

- Some history of science
- Students may be invited to consider the difference between configuration space ($\mathbb{R}^3$), and phase space ($\mathbb{R}^6$ for two-body problem and $\mathbb{R}^9$ for three-body problem).
- Reduction of dimensionality of phase space when constants of the motion are known (e.g., COM and relative coordinates)
- Students see that this is still a vibrant and active area of research.

General Dynamical Systems

- Contrast dynamical systems in physics and mathematics
- State space M
- Evolution rule $\Phi_t : M \rightarrow M$
- Time t
  - Continuous time - differential equation - flow
  - Discrete time - difference equation - map
- Physics students will have seen Newton's equations
- Prior course in differential equations
- Stretch for them to see relationship to simple maps
- Too simple to be useful?

Example: The Bernoulli map

- Discrete time dynamical system on the state space $[0, 1]$
- Dynamical equation: $x_{n+1} = 2x_n \big|_{\bmod 1}$
- Example periodic orbit: $\frac{1}{3} \leftrightarrow \frac{2}{3}$
- Unstable: $0.33 \rightarrow 0.66 \rightarrow 0.32 \rightarrow 0.64 \rightarrow \cdots$ (Yorke)
- Map shifts base-two decimal point to the right
- Any rational number is a periodic point
- Rational numbers countable and dense in $[0, 1]$
- UPOs can be ordered by period
- Irrational numbers give chaotic trajectories

Example: Bernoulli map UPOs

Period	Lyndon word	Initial condition	Observables	
T_p	L_p	x_0	$\sum_{j=0}^{T_p-1} x_j$	$\sum_{j=0}^{T_p-1} x_j^2$
1	0	$.\overline{0} = 0$	0	0
	1	$.\overline{1} = 1$	1	1
2	01	$.\overline{01} = 1/3$	1	$5/9$
3	001	$.\overline{001} = 1/7$	1	$3/7$
	011	$.\overline{011} = 3/7$	2	$10/7$
4	0001	$.\overline{0001} = 1/15$	1	$17/45$
	0011	$.\overline{0011} = 1/5$	2	$6/5$
	0111	$.\overline{0111} = 7/15$	3	$107/45$
5	00001	$.\overline{00001} = 1/31$	1	$11/31$
	00101	$.\overline{00101} = 5/31$	2	$30/31$
	00011	$.\overline{00011} = 3/31$	2	$34/31$
	01011	$.\overline{01011} = 11/31$	3	$61/31$
	00111	$.\overline{00111} = 7/31$	3	$65/31$
	01111	$.\overline{01111} = 15/31$	4	$104/31$

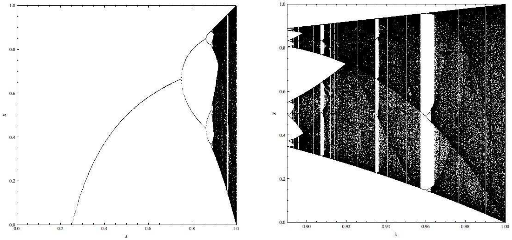
Lessons learned

- Very simple dynamical systems can exhibit both periodicity and chaos
- State space is generally replete with UPOs

Logistic map

- State space is again $[0, 1]$
- One-parameter family of maps:

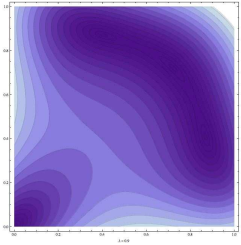
$$x_{n+1} = f_\lambda(x_n) := 4\lambda x_n \left(1 - x_n\right).$$



- Pitchfork bifurcation, stable and unstable periodic orbits
- Easy programming exercise for students

Periodic Orbits of Logistic Map

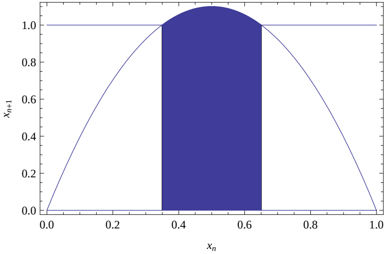
- Finding period-two orbits:
  - Solve: $x = f_\lambda(f_\lambda(x))$
  - Minimize: $F(x,y) = [x - f_\lambda(y)]^2 + [y - f_\lambda(x)]^2$



- Finding period-three orbits, etc:
  - Solve: $x = f_\lambda(f_\lambda(f_\lambda(x)))$
  - Minimize:
    $F(x,y,z) = [x - f_\lambda(y)]^2 + [y - f_\lambda(z)]^2 + [z - f_\lambda(x)]^2$

Cantor Set

- Logistic map with $\lambda > 1$
- Interval leaves state space in one iteration
- Preimages of interval leave in two iterations, etc.



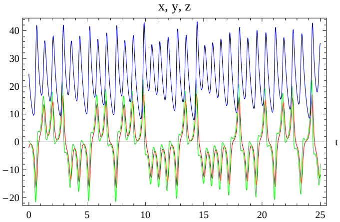
Lessons Learned

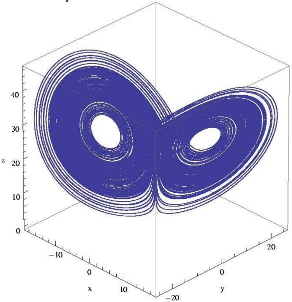
- The notions of transient behavior and attracting set
- Period doubling route to chaos
- Pitchfork bifurcation
- UPOs discovered by either root finding or minimization
- Cantor-set nature of state space when $\lambda > 1$

Lorenz attractor

- Continuous time dynamical system on the state space $\mathbb{R}^3$
- Dynamical equations:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= -xz + Rx - y\\ \dot{z} &= xy - bz\end{aligned}$$

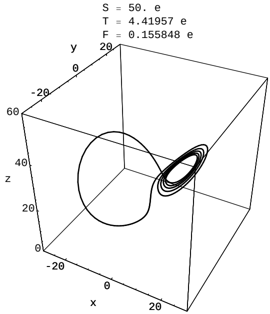




- Attracting set has periodic orbits

Unstable Periodic Orbits of Lorenz Attractor

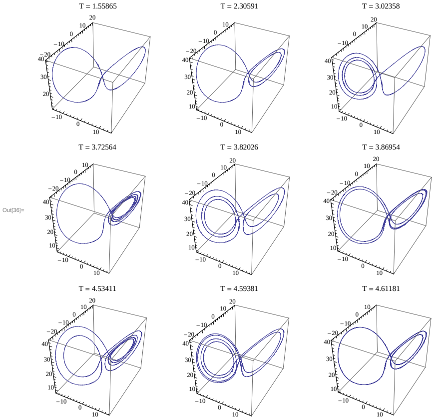
- Correspond to any binary sequence (e.g., 11110)
- Dense in the attractor
- System is *hyperbolic*



- If you know all UPOs with period $< T$, you can make statistical predictions of any observable (DZF formalism).
- UPOs and their properties can be tabulated, stored, and made available in a curated database.

These can be tabulated. . .

- Viswanath, *Nonlinearity* **16** (2003) 1035-1056



- Tabulated up to ~ 20 symbols

Lessons Learned

- These observations work for continuous-time dynamical systems
- The same labeling of orbits used in the Bernoulli map works for the Lorenz attractor
- Symbolic dynamics

Stable periodic orbits

- Laminar (stationary) flow is a fixed point in function space
- von Kármán vortex street is a closed periodic orbit in function space

Turbulent flow in two dimensions

- Reference: N.T. Ouellette, J.P. Gollub, "Curvature Fields, Topology, and the Dynamics of Spatiotemporal Chaos," *Phys. Rev. Lett.* **99** (2007) 194502.
- Periodic flow in square domain
- Periodic force
  $\mathbf{F} = A \sin(2\pi mx) \sin(2\pi ny)\, \mathbf{e}_x + A \cos(2\pi mx) \cos(2\pi ny)\, \mathbf{e}_y$
- Flow closely follows **F** for low Re
- Turbulent for high Re

Lessons Learned

- All this can be made to work for dynamical systems on infinite-dimensional state-spaces

Unstable Periodic Orbits (UPOs)

- Attracting sets in a wide variety of dynamical systems are replete with periodic orbits
- If the dynamics are hyperbolic, the UPOs are unstable
- The UPOs are dense in the attracting set
- The UPOs are countable and have measure zero in the attracting set
- In spite of zero measure, UPOs are exceedingly important, as averages over the natural measure can be derived from them
- "The skeleton of chaos" (Cvitanovic)

Attracting sets and turbulent averages

- The driven Navier-Stokes equations in the turbulent regime describe nonlinear dynamics in an infinite-dimensional (function) space
- These dynamics possess an attracting set
- The attracting set is finite-dimensional, and its dimension grows as a power law in Reynolds number (Constantin, Foias, Manley, Temam, 1985)
- Long-time averages over this attracting set impart a "natural measure" to it
- The problem of turbulence is that of extracting averages of observables over this natural measure

Computing UPOs I: Shooting Method

- Begin on surface of codimension one in function space
- Evolve NS equations until return to that surface
- Use Newton-Raphson to close the gap
- Serial in time
- Constructed to obey equations of motion, but not periodic

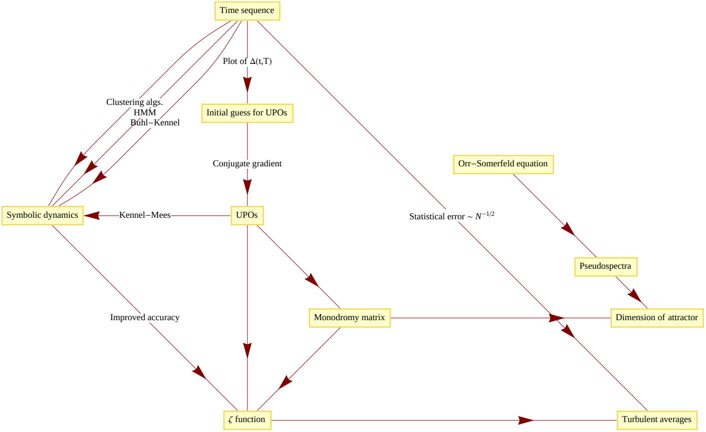
Computing UPOs II: Relaxation Method

- Begin with periodic orbit that is smooth
- Relax to solution of NS equations using variational principle

$$\Delta([f],T) = \frac{1}{2}\sum_{t=0}^{T-1}\sum_{\mathbf{r}}\sum_{j}\left| f_j(\mathbf{r},t+1) - f_j(\mathbf{r},t) - \frac{1}{\tau}\left[f_j^{\mathrm{eq}}(\mathbf{r},t) - f_j(\mathbf{r},t)\right]\right|^2$$

- Constructed to be periodic, but not obey equations of motion
- Conjugate-gradient algorithm
- Higher-order differencing needed
- Local spline fitting to orbit needed
- *Enormous* amounts of memory are needed

Work in Progress I



Work in Progress II

- LUPO: Laboratory for unstable periodic orbits
- Shell models of turbulence (Tang, Boghosian)
- 2D Navier-Stokes (Lätt, Smith, Boghosian)
- 3D Navier-Stokes (Faizendeiro, Coveney, Boghosian)

Conclusions

- UPOs are a unifying concept in dynamical systems
- Connects mathematical and physical way of understanding dynamical systems
- Improved understanding of fluid UPOs may lead to new statistical descriptions of turbulence
- HPC is just at the point where this can be done for infinite-dimensional systems
- LUPO will give students a way to experiment with such systems
- Creation of a UPO database will help spread and share information about UPOs